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BEYOND THE MAGNETIC FIELD
action of forces between moving bodies

Many years ago while I was studying physics in Padua and in particular during the Physics II course I came across the fact that the magnetic force between moving charges does not respect the third law of dynamics. The fact was quickly obscured, in most of the texts by me consulted, attributing the conservation of momentum to the action of the field between particles. Nothing that was then ever demonstrated, often invoking quantum concepts and/or relativistic effects that would then have intervened in more complex theories not possible to be exposed in the texts subject of consultation. On the other hand it is also true that the field effect should also be observed in other forces such as the typically electric one and not only in the magnetic one and it is also true that it is the field itself that gives the force effect either we speak of conservation of momentum referring to the field or we speak of its conservation as a consequence of the behavior of the forces induced by the ca mpo. Otherwise it would not be legitimate to assume Newton?s third law to be valid simply because it would not be true that for every action exercised by guy on guy, guy would react on guy in the opposite way reserving the right to pour the missing part into the surrounding space (on the field itself?) and therefore possibly on other bodies. What I propose in this article (sketched in the now distant 1989) revised in 2022 and decided to share now is to show how the magnetic effect between two moving electrically charged bodies is nothing more than a part of the interaction of the electric force with the speeds of the interacting bodies. The magnetic action of one of the two bodies on the other depends first on the cross product between the speed of one and the electric force and then on the cross product between what results and the speed of the other. The vice versa is also true between the other and the one. However, since the two results are not necessarily opposite, it would seem that for the magnet interactions that Newton?s third law does not hold unless one considers the introduction of another force contribution that appears in Jacobi?s identity concerning the vector product between three vector quantities such as the two velocities of interacting bodies and the electric force among them.

1 Force Between Moving Bodies

the figure 1 refers to the Lorentz force between two moving charged particles of

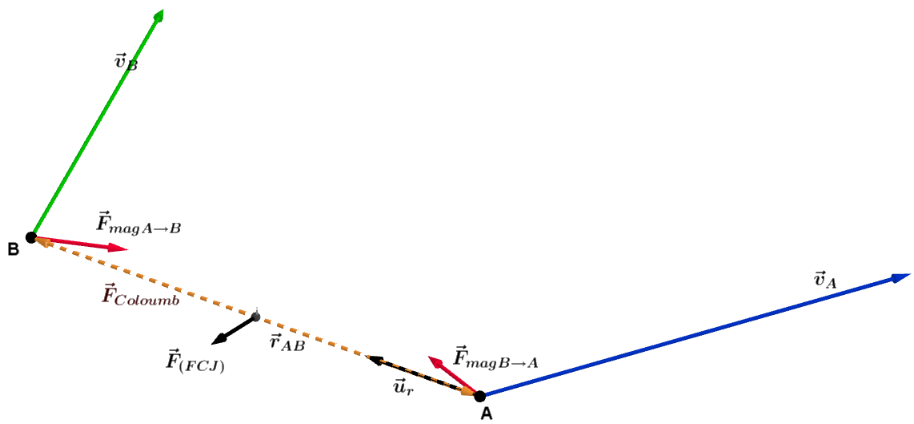

Figure 1: case with coplanar and perpendicular velocities

the same sign.
If we consider the action of A on B, the magnetic component of the Lorentz force is:

$$\vec{F}_{M_{AB}} = \frac{\mu_0 e^2}{4\pi r_{AB}^2} \cdot (\vec{u}_r \times \vec{v}_A) \times \vec{v}_B \tag{1.0.1}$$

where $\vec{r}_{AB} = AB \cdot \vec{u}_r$ and "$\times$" denotes the vector product.
Considering that $\mu_0 = \epsilon_0 / c^2$ the (1.0.1) becomes

$$\vec{F}_{M_{AB}} = \frac{e^2}{4\pi\epsilon_0 r_{AB}^2} \cdot \left(\vec{u}_r \times \frac{\vec{v}_A}{c}\right) \times \frac{\vec{v}_B}{c} = \left(\vec{F}_{AB} \times \frac{\vec{v}_A}{c}\right) \times \frac{\vec{v}_B}{c} \tag{1.0.2}$$

where $\vec{F}_{AB} = \dfrac{\mu_0 e^2}{4\pi r_{AB}^2}\vec{u}_r$ is the purely electric or Coulomb force that A exerts on B[1] . On the opposite side the Lorentz force that B exerts on A is:

$$\vec{F}_{M_{BA}} = \frac{e^2}{4\pi\epsilon_0 r_{BA}^2} \cdot \left(\vec{u}_r \times \frac{\vec{v}_B}{c}\right) \times \frac{\vec{v}_A}{c} = \left(\vec{F}_{BA} \times \frac{\vec{v}_B}{c}\right) \times \frac{\vec{v}_A}{c} \tag{1.0.3}$$

The (1.0.2) and the (1.0.3) represent two of the cross product permutations be-

[1]The expression of the electric and magnetic fields produced by a moving charge can be deduced either from special relativity using Lorentz transformations or from the theory of "Potentials of Liénard-Wiechert".

tween the vectors $\vec{v}_A$, $\vec{v}_B$ e $\vec{F}_{AB}$, bound to the Jacobi identity:

$$\left(\vec{F}_{AB} \times \frac{\vec{v}_A}{c}\right) \times \frac{\vec{v}_B}{c} + \left(\frac{\vec{v}_B}{c} \times \vec{F}_{AB}\right) \times \frac{\vec{v}_A}{c} + \left(\frac{\vec{v}_A}{c} \times \frac{\vec{v}_B}{c}\right) \times \vec{F}_{AB} = 0 \tag{1.0.4}$$

which taking into account the antisymmetry of the vector product and Newton's third law valid for the coulombian forcea $\vec{F}_{AB} = -\vec{F}_{BA}$ can be written:

$$\left(\vec{F}_{AB} \times \frac{\vec{v}_A}{c}\right) \times \frac{\vec{v}_B}{c} + \left(\vec{F}_{BA} \times \frac{\vec{v}_B}{c}\right) \times \frac{\vec{v}_A}{c} + \left(\frac{\vec{v}_A}{c} \times \frac{\vec{v}_B}{c}\right) \times \vec{F}_{AB} = 0 \tag{1.0.5}$$

symmetrical with respect to the indices A and B

$$\begin{aligned} &\left(\vec{F}_{AB} \times \frac{\vec{v}_A}{c}\right) \times \frac{\vec{v}_B}{c} + \left(\vec{F}_{BA} \times \frac{\vec{v}_B}{c}\right) \times \frac{\vec{v}_A}{c} + \\ &\qquad \frac{1}{2}\left(\frac{\vec{v}_B}{c} \times \frac{\vec{v}_A}{c}\right) \times \vec{F}_{BA} + \frac{1}{2}\left(\frac{\vec{v}_A}{c} \times \frac{\vec{v}_B}{c}\right) \times \vec{F}_{AB} = 0 \end{aligned} \tag{1.0.6}$$

The first term of (1.0.6) is the Lorentz force that A exerts on B, the second the Lorentz force that B exerts on A. If we consider only these two terms, at least at a classical level, the third law Newton's is respected only if the velocities are co-directional. In order to reaffirm the momentum conservation principle in general terms, it is necessary to consider the third and fourth term which for the moment we will call the Jacobian completion force (FCJ)..
In tensor notation:

$$Б_B^{'\rho} = \left(\left(\vec{F}_{AB} \times \frac{\vec{v}_A}{c}\right) \times \frac{\vec{v}_B}{c}\right)^\rho = \epsilon_{\alpha\beta\gamma} F^\beta v_A^\gamma \epsilon^{\rho\alpha\delta} v_{B\delta} = \epsilon^{\rho\alpha\delta} \epsilon_{\alpha\beta\gamma} v_A^\gamma v_{B\delta} F^\beta \tag{1.0.7}$$

$$Б_B^{''\rho} = \left(\left(\frac{\vec{v}_A}{c} \times \frac{\vec{v}_B}{c}\right) \times \vec{F}_{AB}\right)^\rho = \epsilon_{\alpha\beta\gamma} v_A^\beta v_B^\gamma \epsilon^{\rho\alpha\delta} F_\delta = \epsilon_{\alpha\beta\gamma} \epsilon^{\rho\alpha\delta} v_A^\beta v_B^\gamma F_\delta \tag{1.0.8}$$

where $\epsilon_{\alpha\beta\gamma}$ is the fully antisymmetric Ricci tensor of the third order and where the Einstein convention on index contraction is used.

$Б_B^{'\rho}$ e $Б_B^{''\rho}$ are respectively the Lorentz force eq.(1.0.3) and the completion force present as the last term in eq.(1.0.5). Considering that:

$$\epsilon_{ijk}\epsilon^{imn} = \begin{vmatrix} \delta_j^m & \delta_j^n \\ \delta_k^m & \delta_k^n \end{vmatrix} = \delta_j^m \delta_k^n - \delta_j^n \delta_k^m \tag{1.0.9}$$

With the use of tensor calculus it is easier to arrive at the following relations:

$$\left(\vec{F}_{AB} \times \frac{\vec{v}_A}{c}\right) \times \frac{\vec{v}_B}{c} = \vec{v}_A \cdot \left(\vec{v}_B \circ \vec{F}_{AB}\right) - \vec{F}_{AB} \cdot (\vec{v}_A \circ \vec{v}_B) \tag{1.0.10}$$

$$\left(\vec{F}_{BA} \times \frac{\vec{v}_B}{c}\right) \times \frac{\vec{v}_A}{c} = \vec{v}_B \cdot \left(\vec{v}_A \circ \vec{F}_{BA}\right) - \vec{F}_{BA} \cdot (\vec{v}_B \circ \vec{v}_A) \tag{1.0.11}$$

$$\left(\frac{\vec{v}_A}{c} \times \frac{\vec{v}_B}{c}\right) \times \vec{F}_{AB} =$$

$$[\vec{v}_B \cdot \left(\vec{F}_{AB} \circ \vec{v}_A\right) - \vec{v}_A \cdot \left(\vec{F}_{AB} \circ \vec{v}_B\right)] =$$

$$[\vec{v}_B \cdot \left(\vec{F}_{AB} \circ \vec{v}_A\right) + \vec{v}_A \cdot \left(\vec{F}_{BA} \circ \vec{v}_B\right)] =$$

$$[-\vec{v}_B \cdot \left(\vec{F}_{BA} \circ \vec{v}_A\right) - \vec{v}_A \cdot \left(\vec{F}_{AB} \circ \vec{v}_B\right)] \tag{1.0.12}$$

where "$\circ$" denotes the scalar product.

The (1.0.10) describes the Lorentz force that A exerts on B, the (1.0.11) describes the Lorentz force that B exerts on A. The first term of the (1.0.12) quantifies the completion force that B exerts on A and the second term of (1.0.12) quantifies the completion force that A exerts on B.

In conclusion, the extra-electric force on B reduces to:

$$\vec{F}_{M_{AB}} = \vec{v}_A \cdot \left(\vec{v}_B \circ \vec{F}_{AB}\right) - \vec{F}_{AB} \cdot (\vec{v}_A \circ \vec{v}_B) - \vec{v}_A \cdot \left(\vec{F}_{AB} \circ \vec{v}_B\right) = -\vec{F}_{AB} \cdot (\vec{v}_A \circ \vec{v}_B) \tag{1.0.13}$$

which, added to the action of the electric field, gives a total action:

$$\vec{F}_{TOT_{AB}} = \vec{F}_{AB} - \vec{F}_{AB} \cdot (\vec{v}_A \circ \vec{v}_B) = \vec{F}_{AB} \cdot (1 - \vec{v}_A \circ \vec{v}_B) \tag{1.0.14}$$

If indeed the total force were reduced to (1.0.14), the conservation of the momentum would be ensured without resorting to intermediate field actions.

2 Second Law of Dynamics Modified

In the light of what has been observed previously, the laws of dynamics could be reformulated: **WHEN A FIELD IS INDUCED BY A MOVING BODY AND ACTS IN A MOBILE POINT ITS GLOBAL EFFECT IS INFLUENCED BY THE INTERACTION BETWEEN THE SPEED OF WHO GENERATES THE FIELD AND THAT OF THE POINT INTERACTING WITH THE FIELDO**

$$\vec{F_{eff}} = \vec{F} - \vec{F} \cdot \left(\frac{\vec{v}_A}{c} \circ \frac{\vec{v}_B}{c} \right) = \vec{F} \cdot \left(1 - \frac{\vec{v}_A}{c} \circ \frac{\vec{v}_B}{c} \right) \tag{2.0.1}$$

where $\vec{v}_A/c$ and $\vec{v}_B/c$ are respectively the normalized speeds of the field generator and of the one interacting with the field.

We could be more daring by modifying the equation of Newtonian dynamics by proposing it again in the form:

$$\frac{m_0}{\sqrt{(1 - v^2/c^2)}} \vec{a} = \vec{F} \left(1 - \frac{\vec{v}_A}{c} \circ \frac{\vec{v}_B}{c} \right) \tag{2.0.2}$$

with acquired meaning of the symbols assuming that the inertia of a body is

$$\frac{m_0}{\sqrt{(1 - v^2/c^2)}}$$

regardless of the direction of action of the force.

The equation (2.0.2) suggests that the combined action of speeds acts on the action of force itself.

3 Electric Field From Moving Charge

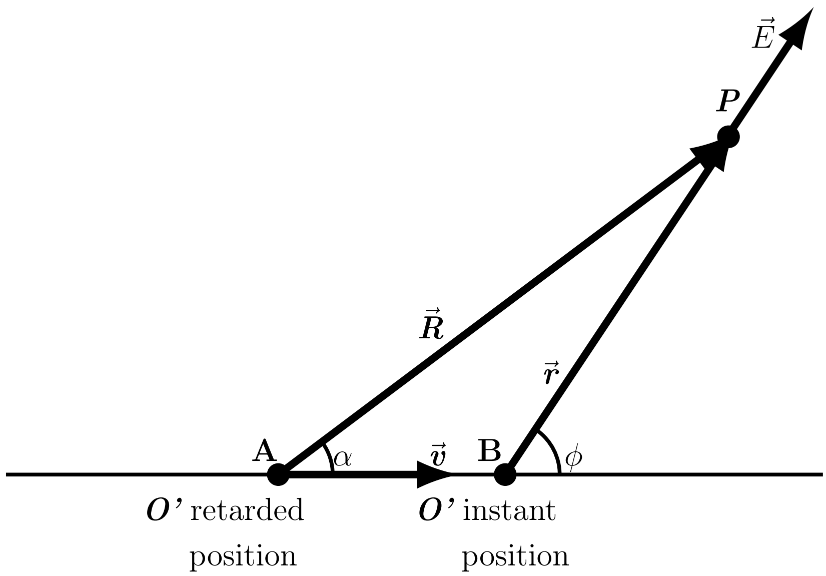

Figure 2: Field $\vec{E}$ Generated By Moving Charge

The quantification of the field generated by a moving charge is reported, as noted by **O** integral with the direction along which it moves $\mathbf{O}'$ sympathetic to the charge, deduced from the "Liénard-Wiechert" theory of retarded potentials (theory on the web by typing the names of the protagonists)

$$\vec{E}(\vec{r},t) = \frac{e}{4\pi\epsilon_0} \frac{\left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \frac{v^2}{c^2}\sin\phi\right)^{3/2}} \frac{\vec{r}}{r^3} \tag{3.0.1}$$

along the abscissa axis, $\sin\phi = 0$:

$$\vec{E}(x,t) = \frac{e}{4\pi\epsilon_0 x^2}\left(1 - \frac{v^2}{c^2}\right) = \frac{e}{4\pi\epsilon_0 x'^2} = \vec{E}'(x',t') \tag{3.0.2}$$

along the ordinate axis, $\sin\phi = 1$:

$$\vec{E}(y,t) = \frac{e}{4\pi\epsilon_0 y^2}\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{e}{4\pi\epsilon_0 y'^2}\gamma = \vec{E}'(y',t')\gamma \tag{3.0.3}$$

The equation (3.0.2) and the equation (3.0.3) respect the laws of transformation of electric fields foreseen by special relativity: In fact, starting from the electromagnetic tensor

$$\mathbb{E}_{i,j} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{3.0.4}$$

applying the Lorentz transformations we get:

$$\mathbb{E}'_{i,j} = \begin{pmatrix} 0 & -E_x & -\gamma(E_y - vB_z) & -\gamma(E_z + vB_y) \\ E'_x & 0 & -\gamma(B_z - vE_y) & \gamma(B_y + vE_z) \\ \gamma(E_y - vB_z) & \gamma(B_z - vE_y) & 0 & -B_x \\ \gamma(E_z + vB_y) & -\gamma(B_y + vE_z) & B_x & 0 \end{pmatrix} \tag{3.0.5}$$

Based on the assumptions made **O'** observes only the electric field and (3.0.4) becomes:

$$\mathbb{E}'_{i,j} = \begin{pmatrix} 0 & -E'_x & -E'_y & -E'_z \\ E'_x & 0 & 0 & 0 \\ E'_y & 0 & 0 & 0 \\ E'_z & 0 & 0 & 0 \end{pmatrix} \tag{3.0.6}$$

. To determine the fields observed by **O** I apply the inverse Lorentz transformations obtaining:

$$\mathbb{E}_{i,j} = \begin{pmatrix} 0 & -E'_x & -\gamma E'_y & -\gamma E'_z \\ E'_x & 0 & -\gamma v E'_y & -\gamma v E'_z \\ \gamma E'_y & \gamma v E'_y & 0 & 0 \\ \gamma E'_z & \gamma v E'_z & 0 & 0 \end{pmatrix} \tag{3.0.7}$$

according to equations (3.0.2) and (3.0.3).

Comparing (3.0.4) with (3.0.7) we note that **O** also observes magnetic fields along y e z.

4 Equations of Dynamics

From the Lorentzian theory of relativity it is known that the inertia along the axis of motion is different from that along the direction perpendicular to the motion. Let us analyze the two cases.

In the inertial system, instant by instant integral with the charge, the Newtonian equations of dynamics are valid:

$$m_0 a'_x = eE'_x = F'_{Ex} \left(= \frac{\mu_0 e^2}{4\pi r'^2} \right) \tag{4.0.1}$$

in the direction of motion e

$$m_0 a'_y = eE'_y = F'_{Ey} \left(= \frac{\mu_0 e^2}{4\pi r'^2} \right) \tag{4.0.2}$$

in the direction perpendicular to the motion. The electric fields E'_x ed E'_y are those observed at the point where the charge is stationed.

To determine the equations in the system where the charge moves with velocity $\vec{v}$ apply the Lorentz transformation laws obtaining respectively:

$$\frac{m_0}{(1 - v^2/c^2)^{3/2}}\, a_x = F'_{Ex} = \frac{\mu_0 e^2}{4\pi r'^2} = \frac{\mu_0 e^2}{4\pi r^2}(1 - v^2/c^2) = F_{Ex} \tag{4.0.3}$$

and

$$\frac{m_0}{(1 - v^2/c^2)}\, a_y = \frac{(E_y - vB_z)e}{\sqrt{1 - v^2/c^2}} = \frac{F_{Ly}}{\sqrt{1 - v^2/c^2}} = F_{Ey}\sqrt{1 - v^2/c^2} \tag{4.0.4}$$

as $B_z = vE_y$, putting $F_{Ex} = eE_x$, $F_{Ey} = eE_y$ and having indicated with F_{Ly} e F_{Ey} respectively the Lorentz force and only the electric force along y.

5 Two Special Cases

5.1 PROBLEM 1

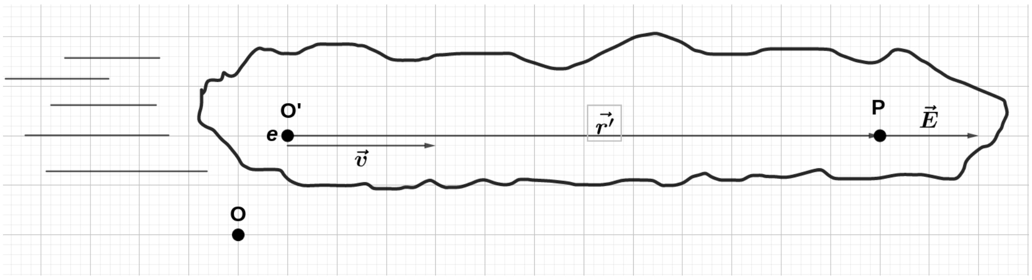

Figure 3: moving charge

Is **O'** the frame of reference in motion with respect to **O** at speed $\vec{v}$ as in fig.3. If in **O'** there is a charge ***e*** at rest, this charge (for the observer $\mathbf{O'}$) generate in **P** an electric field $\vec{E}$**'** directed exclusively along the axis $x' \equiv O'P$ of intensity:

$$E' = \frac{\mu_0 e}{4\pi r'^2} \tag{5.1.1}$$

The invariance of the force for Lorentz transformations along the direction of propagation of the motion is known from the theory of special relativity. In this case total force and field coincide as no magnetic effects are observed.
Let's refer to the law of motion. In the integral reference system a O'

$$m_0 a'_x = eE'_x = F'_{Ex} = \frac{\mu_0 e^2}{4\pi r'^2} \tag{5.1.2}$$

which by applying the Lorentz transformations becomes

$$\frac{m_0}{(1 - v^2/c^2)^{3/2}}\, a_x = F'_{Ex} = \frac{\mu_0 e^2}{4\pi r'^2} = \frac{\mu_0 e^2}{4\pi r^2}(1 - v^2/c^2) = F_{Ex} \tag{5.1.3}$$

or equivalently

$$\frac{m_0}{\sqrt{(1 - v^2/c^2)}}\, a_x = F_{Ex}(1 - v^2/c^2) \tag{5.1.4}$$

where $F_{Ex} = F'_{Ex} = \dfrac{\mu_0 e}{4\pi r'^2}$ is the electric force along x.

The same equation (5.1.4) is obtained by directly applying the equation (2.0.2).

That is, if one wanted to follow a Newtonian approach (all of Einstein's theory is fundamentally a Newton-Lagrangian theory) using, in the formulation of the second law of dynamics (2.0.2) the field calculated with the (3.0.2) we would get the equation directly (5.1.4)

$$\frac{m_0}{(1-v^2/c^2)^{1/2}}\,a_x = F_{Ex}(1-v^2/c^2) = \frac{\mu_0 e}{4\pi r^2}(1-v^2/c^2)(1-v^2/c^2) \qquad (5.1.5)$$

The equation (5.1.5) suggests that the inertia along the direction of motion also remains $m_o\gamma$ being then the combined action of the speeds as described in (2.0.2) to determine a further resistance on the action of the force itself. The energy balance would restore kinetic variations as predicted by the theory of special relativity.

5.2 PROBLEM 2

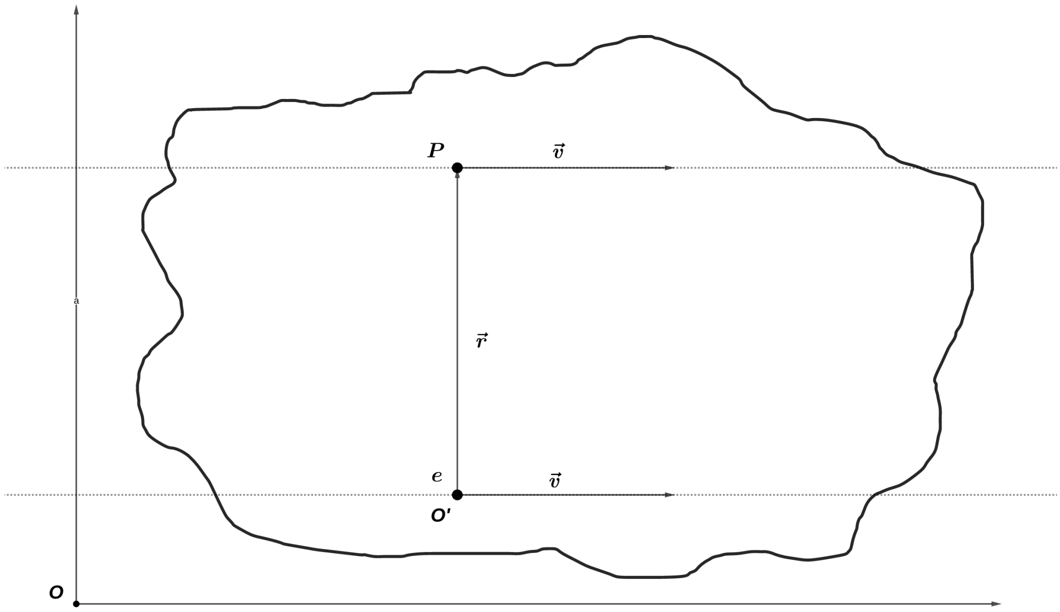

Figure 4: moving charge

Is **O'** the frame of reference in motion with respect to**O** at speed $\boldsymbol{\vec{v}}$ as in fig.4. If in **O'** there is a charge $\boldsymbol{e}$ at rest, this charge (for the observere $\mathbf{O'}$) generate in **P** an electric field $\boldsymbol{\vec{E}}$**'** directed exclusively along the axis $\boldsymbol{y'} \equiv$ **O'P** of intensity:

$$\boldsymbol{E'_y = \frac{e}{4\pi\epsilon_0 r^2}} \tag{5.2.1}$$

being **P** at rest whit respect to **O'**. From equation (3.0.6):

$$\boldsymbol{E_y = \gamma E'_y \qquad B_z = \gamma v E'_y} \tag{5.2.2}$$

$$\boldsymbol{F_{Ey} = \gamma F'_{Ey}, \qquad F_{Ey}\sqrt{(1 - v^2/c^2)} = F'_{Ey}} \tag{5.2.3}$$

where $\boldsymbol{F_{Ey}}$ is the electric force along y axis.
This all yields an overall Lorentz force $\boldsymbol{F_{Ly}}$:

$$\boldsymbol{F_{Ly} = e(E_y - vB_z) = F_{Ey}(1 - v^2/c^2) = F'_{Ey}\sqrt{(1 - v^2/c^2)}} \tag{5.2.4}$$

Finally from equations (5.2.2) and (5.2.4), taking into account that $\boldsymbol{eE'_y} = \boldsymbol{F'_y}$, we have

$$\frac{\boldsymbol{F_{Ly}}}{\sqrt{(1 - v^2/c^2)}} = \boldsymbol{F_{Ey}}\sqrt{(1 - v^2/c^2)} \tag{5.2.5}$$

$$\boldsymbol{F_y} = \boldsymbol{eE_y}(1 - v^2/c^2) \tag{5.2.6}$$

Let us now refer to the equations of motion. In the reference system integral with $\boldsymbol{O'}$

$$\boldsymbol{m_0 a'_y} = \boldsymbol{eE'_y} = \boldsymbol{F'_{Ey}} \tag{5.2.7}$$

applying the Lorentz transformations

$$\frac{\boldsymbol{m_0}}{(1 - v^2/c^2)}\,\boldsymbol{a_y} = \boldsymbol{F_{Ey}}\sqrt{1 - v^2/c^2} = \frac{\boldsymbol{F_{Ly}}}{\sqrt{1 - v^2/c^2}} \tag{5.2.8}$$

Meaning what

$$\frac{\boldsymbol{m_0}}{\sqrt{(1 - v^2/c^2)}}\,\boldsymbol{a_y} = \boldsymbol{F_{Ly}} \tag{5.2.9}$$

where $\boldsymbol{F_{Ly}}$is the Lorentz force along y.

Using, in the definition of Newton's second law eq.(2.0.2), the field defined by (3.0.3), one directly obtains the (5.2.9)

$$\frac{\boldsymbol{m_0}}{\sqrt{(1 - v^2/c^2)}}\,\boldsymbol{a_y} = \boldsymbol{F_{Ey}}(1 - v^2/c^2) \tag{5.2.10}$$

6 Magnetic Action and Reaction

In order to justify the action of the electric field on speed, we might have to accept that in space-time the concepts of point, line and surface are purely mathematical and not real and that this is also value for time. a four-volume element must be determined by intervals (not punctual) in all its coordinates. in our reality there will only exist four-volume elements with finite dimensions.

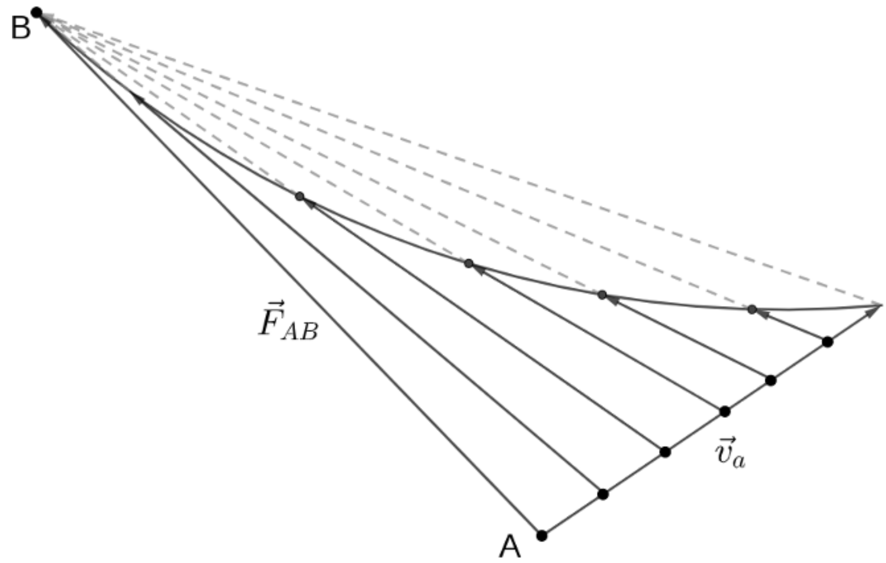

Figure 5: straight rotation of action of force

THE MOVEMENT WILL TAKE PLACE ALONG A CONTINUOUS TIME DIMENSION BUT YOU MOVE "A SALTINI" SIMULTANEOUSLY OCCUPYING ALL THE INSTANTS OF THE INDIVISIBLE QUANTITY OF TIME PAST INSTANTLY AND ALL THE POSITIONS WITHIN THE CORRESPONDING SPACIAL INTERVAL PATH PROPORTIONAL TO THE SPEED OF WHOM YOU ARE OBSERVING. In this way the force does not act from a precise point but we will be forced to deal with bundles of force (not necessarily parallel as in figure (5). The force would be able to impose a moment being able to act along the arm $\boldsymbol{\vec{v}dt}$ deviating from the effective line of action AB.

Let us not forget that this is inherent in the measurement procedure since it is not possible to measure the speed of the body at a point in space except in reference to a limit mathematical procedure which leads to results with consequences that can be verified by measurement but not directly verifiable at the point, bringing us as stated in paragraph 1.

It could be hypothesized that the field information also reports the "absolute instant" in which it was emitted, therefore the principles of conservation will refer to that instant or to that continuous and indivisible time interval. A kind of gauge OTP at work in our universe.